A Paraconsistent Semantics for Generalized Logic Programs

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Outline

Introduction

- Increasing interest in extensions of the logic programs
- What is the intended semantics of such programs?
- One suggestion are the stable generated models of [HW97]
- Problem: a local inconsistency trivialize the whole program
- We define a three-valued paraconsistent semantics which extends the stable generated models

Preliminaries I

- signature $\sigma = \langle Rel, Const, Fun \rangle$
- $\mathrm{At}(\sigma)$ the set of all atomic formulas
- $L(\sigma)$ is defined inductively:
  1. $\mathrm{At}(\sigma) \subseteq L(\sigma)$
  2. If $F, G \in L(\sigma)$,
     then $\{\neg F,\ F \wedge G,\ F \vee G,\ F \rightarrow G,\ \exists xF, \forall xF\} \subseteq L(\sigma)$
- $L^0(\sigma)$ denotes the corresponding set of *sentences*
- Let $\overline{X} \subseteq L(\sigma)$, then $\overline{X} = \{\neg F \mid F \in X\}$
- $\mathrm{Lit}(\sigma) = \mathrm{At}(\sigma) \cup \overline{\mathrm{At}(\sigma)}$ the set of all *literals*

Preliminaries II

Definition (Herbrand Interpretation)

A Herbrand σ-interpretation is a set of literals $I \subseteq \mathrm{Lit}^0(\sigma)$ satisfying the condition $\{a, \neg a\} \cap I \neq \emptyset$ for every ground atom $a \in \mathrm{At}^0(\sigma)$.

- $\mathbf{I_H}(\sigma)$ denotes the class of all Herbrand σ-interpretations
- I can be represented as a function from $At^0(\sigma)$ to $\{t, f, \top\}$
  1. $I(a) = \top$, if $\{a, \neg a\} \subseteq I$
  2. $I(a) = t$, if $a \in I$ and $\neg a \notin I$
  3. $I(a) = f$, if $a \notin I$ and $\neg a \in I$
- linear order $f < \top < t$
- function neg : $neg(t) = f, neg(f) = t, neg(\top) = \top$

Preliminaries III

Definition (Model Relation)

The mapping $\overline{I} : L(\sigma) \rightarrow \{t, f, \top\}$ is defined inductively by the following conditions:

1. $\overline{I}(F) = I(F)$ for every $F \in At^0(\sigma)$
2. $\overline{I}(\neg F) = neg(\overline{I}(F))$
3. $\overline{I}(F \wedge G) = min\{\overline{I}(F), \overline{I}(G)\}$
4. $\overline{I}(F \vee G) = max\{\overline{I}(F), \overline{I}(G)\}$
5. $\overline{I}(F \rightarrow G) = \overline{I}(\neg F \vee G)$
6. $\overline{I}(\exists x F(x))= sup\{\overline{I}(F(x/t)) \ : \ t \in U(\sigma)\}$
7. $\overline{I}(\forall x F(x))= inf\{\overline{I}(F(x/t)) \ : \ t \in U(\sigma)\}$

Preliminaries IV

- the set of designated truth values: $\{t, \top\}$, i.e. $I \models F$ iff $\overline{I}(F) \in \{t, \top\}$ for $I \in \mathbf{I_H}(\sigma)$ and $F \in L^0(\sigma)$
- $I \models F$ iff $I \models \nu(F)$ for every valuation ν and $F \in L(\sigma)$
- Herbrand model operator: $\mathrm{Mod}(X) = \{I \in \mathbf{I_H}(\sigma) : I \models X\}$
- corresponding consequence relation: $X \models F$ iff $\mathrm{Mod}(X) \subseteq \mathrm{Mod}(F)$ for $X \subseteq L(\sigma)$

Proposition ([We98])

The consequence operator C defined by $C(X) = \{F \mid X \models F\}$ is not conservative.

Minimal Models I

Let I be an Herbrand interpretation, we define

- $Pos(I) = I \cap At^0(\sigma)$
- $Neg(I) = I \cap \{\neg a : a \in At^0(\sigma)\}$
- $inc(I) = \{a : \{a, \neg a\} \subseteq I\}$

Definition

Let I, J be Herbrand interpretations. Then we define

1. $I \preceq J$ iff $Pos(I) \subseteq Pos(J)$ and $Neg(J) \subseteq Neg(I)$
2. $I \sqsubseteq J$ iff $inc(I) \subseteq inc(J)$.

Minimal Models II

Definition

Let X be a set of formulas and *I* be an interpretation.

1. I is a t-minimal model of X iff $I \in \mathrm{Min}_{\preceq}(\mathrm{Mod}(X))$.
2. I is an inc-minimal model of X iff $I \in \mathrm{Min}_{\sqsubseteq}(\mathrm{Mod}(X))$.

Proposition

Let T be a quantifier-free theory and I an inc-minimal model of the theory T. Then there exists a model J of T such that

1. $inc(I) = inc(J)$
2. $J \preceq I$
3. *for all* $J_0 \preceq J$ *such that* $J_0 \neq J$ *either* $inc(J_0) \neq inc(I)$ *or* $J_0 \not\models T$.

Sequents and Logic Programs I

Definition (Sequent)

A *sequent* s is an expression of the form:

$$F_1, \ldots, F_m \Rightarrow G_1, \ldots, G_n$$

where $F_i, G_j \in L(\sigma)$ for $i = 1, \ldots, m$ and $j = 1, \ldots, n$.

- *Body* of s: $B(s) = \{F_1, \ldots, F_m\}$
- *Head* of s: $H(s) = \{G_1, \ldots, G_n\}$
- $\mathrm{Seq}(\sigma)$: the class of all sequents s with $H(s), B(s) \subseteq L(\sigma)$
- $[S]$: set of all ground instances of sequences from $S \subseteq \mathrm{Seq}(\sigma)$

Sequents and Logic Programs II

Definition (Model of a Sequent)

Let $I \in \mathbf{I_H}$. Then, $I \models F_1, \ldots, F_m \Rightarrow G_1, \ldots, G_n$ iff for all ground substitutions the following condition is satisfied:
$I \models \bigwedge_{i \leq m} \nu(F_i) \rightarrow \bigvee_{j \leq n} \nu(G_j)$.
Then I is said to be a model of $F_1, \ldots, F_m \Rightarrow G_1, \ldots, G_n$.

Definition (Classes of Logic Programs)

1. Normal Logic Program
   $\mathrm{NLP}(\sigma) = \{s \in \mathrm{Seq}(\sigma) : H(s) \in \mathrm{At}(\sigma),\ B(s) \subseteq \mathrm{Lit}(\sigma)\}$
2. Generalized Logic Program
   $\mathrm{GLP}(\sigma) = \{s \in \mathrm{Seq}(\sigma) : H(s), B(s) \subseteq L(\sigma; \neg, \wedge, \vee, \rightarrow)\}$

Sequents and Logic Programs III

Definition (Inc-t-minimal Model)

A model I of $P \subseteq \mathrm{GLP}(\sigma)$ is said to be *inc-t-minimal* if I is *inc-minimal* and there is no model J of P satisfying the conditions $inc(J) = inc(I)$, $J \preceq I$, $J \neq I$.

Example

$P = \{\Rightarrow r(c); \Rightarrow \neg p(a); \Rightarrow \neg p(b); \Rightarrow p(a), p(b); \neg p(x) \Rightarrow q(x)\}$.
Every intended model of P should contain q(c).
But there exists an inc-t-minimal model of P:

$$M_1 = \{ \neg p(a), p(a), \neg p(b), p(c), q(a), q(b), \neg q(c), \neg r(a), \neg r(b), r(c)\}$$

Paraconsistent Stable Generated Models I

Definition (Interpretation Interval)

Let $I_1, I_2 \in \mathbf{I_H}(\sigma)$ such that $inc(I_1) = inc(I_2)$.
$[I_1, I_2] = \{I \in \mathbf{I_H}(\sigma) : I_1 \preceq I \preceq I_2 \text{ and } inc(I) = inc(I_1)\}$.
For $P \subseteq \mathrm{GLP}(\sigma)$ let be $P_{[I_1,I_2]} = \{r \mid r \in [P] \text{ and } [I_1, I_2] \models B(r)\}$.

Definition (Paraconsistent Stable Generated Model)

Let $P \subseteq \mathrm{GLP}(\sigma)$. An inc-minimal model M of P is called *paraconsistent stable generated*, symbolically $M \in \mathrm{Mod}_{ps}(P)$, if there is a chain of Herbrand interpretations $I_0 \preceq \ldots \preceq I_\kappa$ such that $M = I_\kappa$, and

Paraconsistent Stable Generated Models II

Definition (Paraconsistent Stable Generated Model)

1. M is inc-minimal
2. $I_0 = inc(M) \cup \{\neg a \mid a \in At^0(\sigma)\}$.
3. For successor ordinals α with $0 < \alpha \leq \kappa$, I_α is a $\preceq$-minimal extension of $I_{\alpha-1}$ satisfying the heads of all sequents whose bodies hold in $[I_{\alpha-1}, M]$, i.e.

$$I_\alpha \in \mathrm{Min}_{\preceq}\{\, I \in \mathbf{I_H}(\sigma) : M \succeq I \succeq I_{\alpha-1}, inc(M) = inc(I), \\ I \models \bigvee H(s), \text{ for all } s \in P_{[I_{\alpha-1},M]} \,\}$$

We also say that M is *generated* by the *P-stable chain* $I_0 \preceq \ldots \preceq I_\kappa$.

Paraconsistent Stable Generated Models III

Example

$P = \{\Rightarrow r(c); \Rightarrow \neg p(a); \Rightarrow \neg p(b); \Rightarrow p(a), p(b); \neg p(x) \Rightarrow q(x)\}$.
Because of the rules $\{\Rightarrow \neg p(a); \Rightarrow \neg p(b); \Rightarrow p(a), p(b)\}$ it is easy to see that P has no two-valued model.
But there are two paraconsistent stable generated models:

$M_1 = \{ \neg r(a), \neg r(b), r(c), \neg p(a), \neg p(b), p(a), \neg p(c),$
$q(a), q(b), q(c)\}$ and
$M_2 = \{\neg r(a), \neg r(b), r(c), \neg p(a), \neg p(b), p(b), \neg p(c),$
$q(a), q(b), q(c)\}$.

The model M_1 is constructed by the chain $I_0^1 \preceq I_1^1 = M_1$.

$$I_0^1 = \{p(a)\} \cup \{ \neg r(a), \neg r(b), \neg r(c), \neg p(a), \neg p(b), \neg p(c), \neg q(a), \neg q(b), \neg q(c)\}$$

Paraconsistent Stable Generated Models IV

Example (continuation)

$$P_{[I_0^1,M_1]} = \{ \Rightarrow r(c); \Rightarrow \neg p(a); \Rightarrow \neg p(b); \Rightarrow p(a), p(b); \\ \neg p(a) \Rightarrow q(a); \neg p(b) \Rightarrow q(b); \neg p(c) \Rightarrow q(c)\}$$

$$I_1^1 = M_1 = \{ \neg r(a), \neg r(b), r(c), \neg p(a), \neg p(b), p(a), \neg p(c), \\ q(a), q(b), q(c)\}.$$

The model M_2 is constructed by the chain $I_0^2 \preceq I_1^2 = M_2$.

$$I_0^2 = \{p(b)\} \cup \{ \neg r(a), \neg r(b), \neg r(c), \neg p(a), \neg p(b), \neg p(c), \\ \neg q(a), \neg q(b), \neg q(c)\}.$$

$$P_{[I_0^2,M_2]} = \{ \Rightarrow r(c); \Rightarrow \neg p(a); \Rightarrow \neg p(b); \Rightarrow p(a), p(b); \\ \neg p(a) \Rightarrow q(a); \neg p(b) \Rightarrow q(b); \neg p(c) \Rightarrow q(c)\}$$

$$I_1^2 = M_2 = \{ \neg r(a), \neg r(b), r(c), \neg p(a), \neg p(b), p(b), \neg p(c), \\ q(a), q(b), q(c)\}$$

Paraconsistent Stable Generated Models V

Proposition

Let P be a generalized logic program, and assume P is consistent, i.e. has a two-valued classical interpretation. Then a model I of P is paraconsistent stable generated if and only if it is stable generated (in the sense of [HW97]).

Corollary

Let P be a normal logic program. Then a model I of P is paraconsistent stable generated if and only if it is stable (in the sense of [GL88]).

Conclusion

We propose a paraconsistent semantics which generalizes the notion of the stable generated models to possibly inconsistent logic programs.

References

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