A Paraconsistent Semantics for Generalized Logic Programs

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Introduction

- Increasing interest in extensions of the logic programs
- What is the intended semantics of such programs?
- One suggestion are the stable generated models of [HW97]
- Problem: a local inconsistency trivialize the whole program
- We define a three-valued paraconsistent semantics which extends the stable generated models
Preliminaries I

- signature $\sigma = \langle \text{Rel}, \text{Const}, \text{Fun} \rangle$

- $\text{At}(\sigma)$ the set of all atomic formulas

- $L(\sigma)$ is defined inductively:
  1. $\text{At}(\sigma) \subseteq L(\sigma)$
  2. If $F, G \in L(\sigma)$, then $\{\neg F, F \land G, F \lor G, F \rightarrow G, \exists x F, \forall x F\} \subseteq L(\sigma)$

- $L^0(\sigma)$ denotes the corresponding set of sentences

- Let $\overline{X} \subseteq L(\sigma)$, then $\overline{X} = \{-F \mid F \in X\}$

- $\text{Lit}(\sigma) = \text{At}(\sigma) \cup \overline{\text{At}(\sigma)}$ the set of all literals
Definition (Herbrand Interpretation)

A Herbrand $\sigma$-interpretation is a set of literals $I \subseteq \text{Lit}^0(\sigma)$ satisfying the condition $\{a, \neg a\} \cap I \neq \emptyset$ for every ground atom $a \in \text{At}^0(\sigma)$.

- $I_H(\sigma)$ denotes the class of all Herbrand $\sigma$-interpretations
- $I$ can be represented as a function from $\text{At}^0(\sigma)$ to $\{t, f, \top\}$
  1. $I(a) = \top$, if $\{a, \neg a\} \subseteq I$
  2. $I(a) = t$, if $a \in I$ and $\neg a \not\in I$
  3. $I(a) = f$, if $a \not\in I$ and $\neg a \in I$
- linear order $f < \top < t$
- function $\text{neg} : \text{neg}(t) = f, \text{neg}(f) = t, \text{neg}(\top) = \top$
Definition (Model Relation)

The mapping $\overline{I} : L(\sigma) \rightarrow \{ t, f, \top \}$ is defined inductively by the following conditions:

1. $\overline{I}(F) = I(F)$ for every $F \in At^0(\sigma)$
2. $\overline{I}(\neg F) = \text{neg}(\overline{I}(F))$
3. $\overline{I}(F \land G) = \min\{\overline{I}(F), \overline{I}(G)\}$
4. $\overline{I}(F \lor G) = \max\{\overline{I}(F), \overline{I}(G)\}$
5. $\overline{I}(F \rightarrow G) = \overline{I}(\neg F \lor G)$
6. $\overline{I}(\exists x F(x)) = \sup\{\overline{I}(F(x/t)) : t \in U(\sigma)\}$
7. $\overline{I}(\forall x F(x)) = \inf\{\overline{I}(F(x/t)) : t \in U(\sigma)\}$
the set of designated truth values: $\{t, \top\}$,
i.e. $I \models F$ iff $\bar{T}(F) \in \{t, \top\}$ for $I \in I_H(\sigma)$ and $F \in L^0(\sigma)$

$I \models F$ iff $I \models \nu(F)$ for every valuation $\nu$ and $F \in L(\sigma)$

Herbrand model operator: $\text{Mod}(X) = \{I \in I_H(\sigma) : I \models X\}$

corresponding consequence relation: $X \models F$ iff $\text{Mod}(X) \subseteq \text{Mod}(F)$ for $X \subseteq L(\sigma)$

Proposition ([We98])

The consequence operator $C$ defined by $C(X) = \{F \mid X \models F\}$ is not conservative.
Let \( I \) be an Herbrand interpretation, we define

\begin{align*}
\bullet & \quad Pos(I) = I \cap At^0(\sigma) \\
\bullet & \quad Neg(I) = I \cap \{ \neg a : a \in At^0(\sigma) \} \\
\bullet & \quad inc(I) = \{ a : \{ a, \neg a \} \subseteq I \}
\end{align*}

**Definition**

Let \( I, J \) be Herbrand interpretations. Then we define

1. \( I \preceq J \) iff \( Pos(I) \subseteq Pos(J) \) and \( Neg(J) \subseteq Neg(I) \)

2. \( I \sqsubseteq J \) iff \( inc(I) \subseteq inc(J) \).
Minimal Models II

Definition
Let $X$ be a set of formulas and $I$ be an interpretation.
1. $I$ is a t-minimal model of $X$ iff $I \in \text{Min}_{\preceq} (\text{Mod}(X))$.
2. $I$ is an inc-minimal model of $X$ iff $I \in \text{Min}_{\sqsubseteq} (\text{Mod}(X))$.

Proposition
Let $T$ be a quantifier-free theory and $I$ an inc-minimal model of the theory $T$. Then there exists a model $J$ of $T$ such that
1. $\text{inc}(I) = \text{inc}(J)$
2. $J \preceq I$
3. for all $J_0 \preceq J$ such that $J_0 \neq J$ either $\text{inc}(J_0) \neq \text{inc}(I)$ or $J_0 \not\models T$. 

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Sequents and Logic Programs I

Definition (Sequent)

A sequent $s$ is an expression of the form:

$$F_1, \ldots, F_m \Rightarrow G_1, \ldots, G_n$$

where $F_i, G_j \in L(\sigma)$ for $i = 1, \ldots, m$ and $j = 1, \ldots, n$.

- **Body of $s$**: $B(s) = \{F_1, \ldots, F_m\}$
- **Head of $s$**: $H(s) = \{G_1, \ldots, G_n\}$
- **Seq$(\sigma)$**: the class of all sequents $s$ with $H(s), B(s) \subseteq L(\sigma)$
- **$[S]$**: set of all ground instances of sequences from $S \subseteq \text{Seq}(\sigma)$
Sequents and Logic Programs II

Definition (Model of a Sequent)

Let $I \in \mathcal{I}_H$. Then, $I \models F_1, \ldots, F_m \Rightarrow G_1, \ldots, G_n$ iff for all ground substitutions the following condition is satisfied:

$I \models \bigwedge_{i \leq m} \nu(F_i) \rightarrow \bigvee_{j \leq n} \nu(G_j)$.

Then $I$ is said to be a model of $F_1, \ldots, F_m \Rightarrow G_1, \ldots, G_n$.

Definition (Classes of Logic Programs)

1. Normal Logic Program
   $\text{NLP}(\sigma) = \{ s \in \text{Seq}(\sigma) : H(s) \in \text{At}(\sigma), B(s) \subseteq \text{Lit}(\sigma) \}$

2. Generalized Logic Program
   $\text{GLP}(\sigma) = \{ s \in \text{Seq}(\sigma) : H(s), B(s) \subseteq L(\sigma; \neg, \land, \lor, \rightarrow) \}$
Sequents and Logic Programs III

Definition (Inc-t-minimal Model)
A model $I$ of $P \subseteq \text{GLP}(\sigma)$ is said to be $\text{inc-t-minimal}$ if $I$ is $\text{inc-minimal}$ and there is no model $J$ of $P$ satisfying the conditions $\text{inc}(J) = \text{inc}(I)$, $J \preceq I$, $J \neq I$.

Example
$P = \{ \Rightarrow r(c); \Rightarrow \neg p(a); \Rightarrow \neg p(b); \Rightarrow p(a), p(b); \neg p(x) \Rightarrow q(x) \}$. 
Every intended model of $P$ should contain $q(c)$.
But there exists an inc-t-minimal model of $P$: 
$M_1 = \{ \neg p(a), p(a), \neg p(b), p(c), q(a), q(b), \neg q(c), \neg r(a), \neg r(b), r(c) \}$
Paraconsistent Stable Generated Models I

Definition (Interpretation Interval)

Let $l_1, l_2 \in I_H(\sigma)$ such that $inc(l_1) = inc(l_2)$.

$[l_1, l_2] = \{ l \in I_H(\sigma) : l_1 \preceq l \preceq l_2 \text{ and } inc(l) = inc(l_1) \}$.

For $P \subseteq \text{GLP}(\sigma)$ let be $P[l_1, l_2] = \{ r \mid r \in [P] \text{ and } [l_1, l_2] \models B(r) \}$.

Definition (Paraconsistent Stable Generated Model)

Let $P \subseteq \text{GLP}(\sigma)$. An inc-minimal model $M$ of $P$ is called 
paraconsistent stable generated, symbolically $M \in \text{Mod}_{ps}(P)$, if

there is a chain of Herbrand interpretations $l_0 \preceq \ldots \preceq l_\kappa$ such

that $M = l_\kappa$, and
Paraconsistent Stable Generated Models II

Definition (Paraconsistent Stable Generated Model)

1. $M$ is inc-minimal

2. $l_0 = \text{inc}(M) \cup \{ \neg a \mid a \in \text{At}^0(\sigma) \}$.

3. For successor ordinals $\alpha$ with $0 < \alpha \leq \kappa$, $l_\alpha$ is a \leq\text{-minimal extension of $l_{\alpha - 1}$ satisfying the heads of all sequents whose bodies hold in $[l_{\alpha - 1}, M]$, i.e.

$$l_\alpha \in \text{Min}_\leq \{ l \in \text{I}_H(\sigma) : M \succeq l \succeq l_{\alpha - 1}, \text{inc}(M) = \text{inc}(l), l \models \bigvee H(s), \text{for all } s \in P[l_{\alpha - 1}, M] \}$$

We also say that $M$ is generated by the $P$-stable chain

$l_0 \leq \ldots \leq l_\kappa$. 

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Paraconsistent Stable Generated Models III

Example

\( P = \{ \Rightarrow r(c); \Rightarrow \neg p(a); \Rightarrow \neg p(b); \Rightarrow p(a), p(b); \neg p(x) \Rightarrow q(x) \} \).

Because of the rules \( \{ \Rightarrow \neg p(a); \Rightarrow \neg p(b); \Rightarrow p(a), p(b) \} \) it is easy to see that \( P \) has no two-valued model.

But there are two paraconsistent stable generated models:

\[
M_1 = \{ \neg r(a), \neg r(b), r(c), \neg p(a), \neg p(b), p(a), \neg p(c), q(a), q(b), q(c) \} \]

and

\[
M_2 = \{ \neg r(a), \neg r(b), r(c), \neg p(a), \neg p(b), p(b), \neg p(c), q(a), q(b), q(c) \}. \]

The model \( M_1 \) is constructed by the chain \( I_0^1 \preceq I_1^1 = M_1 \).

\[
I_0^1 = \{ p(a) \} \cup \{ \neg r(a), \neg r(b), \neg r(c), \neg p(a), \neg p(b), \neg p(c), \neg q(a), \neg q(b), \neg q(c) \} \]
Example (continuation)

\[ P_{[I_0^1, M_1]} = \{ \Rightarrow r(c); \Rightarrow \neg p(a); \Rightarrow \neg p(b); \Rightarrow p(a), p(b); \neg p(a) \Rightarrow q(a); \neg p(b) \Rightarrow q(b); \neg p(c) \Rightarrow q(c) \} \]

\[ I_1^1 = M_1 = \{ \neg r(a), \neg r(b), r(c), \neg p(a), \neg p(b), p(a), \neg p(c), q(a), q(b), q(c) \}. \]

The model \( M_2 \) is constructed by the chain \( I_0^2 \preceq I_1^2 = M_2 \).

\[ I_0^2 = \{ p(b) \} \cup \{ \neg r(a), \neg r(b), \neg r(c), \neg p(a), \neg p(b), \neg p(c), \neg q(a), \neg q(b), \neg q(c) \}. \]

\[ P_{[I_0^2, M_2]} = \{ \Rightarrow r(c); \Rightarrow \neg p(a); \Rightarrow \neg p(b); \Rightarrow p(a), p(b); \neg p(a) \Rightarrow q(a); \neg p(b) \Rightarrow q(b); \neg p(c) \Rightarrow q(c) \} \]

\[ I_1^2 = M_2 = \{ \neg r(a), \neg r(b), r(c), \neg p(a), \neg p(b), p(b), \neg p(c), q(a), q(b), q(c) \}. \]
Proposition

Let $P$ be a generalized logic program, and assume $P$ is consistent, i.e. has a two-valued classical interpretation. Then a model $I$ of $P$ is paraconsistent stable generated if and only if it is stable generated (in the sense of [HW97]).

Corollary

Let $P$ be a normal logic program. Then a model $I$ of $P$ is paraconsistent stable generated if and only if it is stable (in the sense of [GL88]).
Conclusion

We propose a paraconsistent semantics which generalizes the notion of the stable generated models to possibly inconsistent logic programs.
References

[GL88] Gelfond, M. and Lifschitz, V.:
The stable model semantics for logic programming.

[HW97] Herre, H. and Wagner, G.:
Stable Models Are Generated by a Stable Chain.
Journal of Logic Programming, 30 (2): 166-177, 1997

[We98] Weber, S.:
Investigations in Belnap’s Logic of Inconsistent and
Unknown Information.
Dissertation, University of Leipzig, 1998